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THE PROBLEM OF THE PRESSURE OF A RIGID STAMP ON THE BOUNDARY OF A NON-LINEARLY ELASTIC HALF-PLANE UNDER FINITE DEFORMATIONS*

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The plane contact problem of non-linear elasticity theory is considered for a half-plane of non-linearly elastic material of harmonic type /1/ under finite deformations. It is assumed that there is no friction in the area of stamp contact with the elastic half-plane. The problem is reduced to a non-linear integral equation by using the scheme the author proposed earlier /2/. Unlike /2/, where this equation is solved just for a flat stamp with a rectilinear horizontal base, an exact solution is obtained for an inclined stamp with a flat base as well as for a stamp whose base profile is the arc of a cricle or wedge. It is shown that the contact pressure is bounded at the stamp edges and at the corner point.

1. Formulation and solution of the problem. The physical domain under consideration is the lower half-plane S^- of the plane of the variable $z = x + iy$, and L is the boundary of the domain S^-. We assume that a rigid stamp presses without friction on a part $L_1 = [ab]$ of this boundary. There are no external effects on the remainder of the boundary $L_2 = L \setminus L_1$. The stamp is pressed to the boundary by external forces whose principal vector is $(0, -N_0)$. We will consider the stamp to be displaced transversely only but along the normal to the boundary. There are no stresses and rotation at infinity.

The boundary conditions of the problem have the form /3, 4/

$$X_y = 0 \text{ on } L,\ Y_y = 0 \text{ on } L_2,\ v^- = f(x + u^-) + C \text{ on } L_1 \tag{1.1}$$

where Y_y, X_y are components of the true Cauchy stress tensor, u^-, v^- are boundary values of the displacement components u and v, respectively, on L, f is a real function characterizing the shape of the stamp base profile: $f_x' \in H(L_1)$, and C is an arbitrary real constant. The boundary conditions and the behaviour of the solution of the problem for the case of several stamps are exactly identical to the case for one stamp.

The mentioned limit value of the horizontal displacement $u = u(x)\ (x^* = x + u)$ of points of the contact domain figure in the last equality of condition (1.1). This function is unknown and to be determined when solving the problem, which is also a considerable difference between the problem under investigation and that considered in /2/.

Let us use complex representations of the fields of the elastic elements in terms of two analytic functions $\varphi(z)$ and $\psi(z)$ of the complex argument $z = x + iy$ in the domain S^- under consideration /2, 3/

$$X_x + Y_y + 4\mu = \frac{(\lambda + 2\mu)\, q\Omega(q)}{\sqrt{I}}, \tag{1.2}$$

$$Y_y - X_x - 2iX_y = -\frac{4(\lambda + 2\mu)}{\sqrt{I}} \frac{\Omega(q)}{q} \frac{\partial z^*}{\partial z} \frac{\partial z^*}{\partial \bar{z}}$$

$$u_x' + iv_x' = \frac{\mu}{\lambda + 2\mu} \varphi'^2(z) + \frac{\lambda + \mu}{\lambda + 2\mu} \frac{\varphi'(z)}{\overline{\varphi'(z)}} - \frac{\lambda + \mu}{\lambda + 2\mu} \left[\frac{\varphi(z)\overline{\varphi''(z)}}{\overline{\varphi'^2(z)}} - \overline{\psi'(z)} \right] - 1 \tag{1.3}$$

$$\frac{\partial z^*}{\partial z} = \frac{\mu}{\lambda + 2\mu} \varphi'^2(z) + \frac{\lambda + \mu}{\lambda + 2\mu} \frac{\varphi'(z)}{\overline{\varphi'(z)}}, \quad \frac{\partial z^*}{\partial \bar{z}} = -\frac{\lambda + \mu}{\lambda + 2\mu} \left[\frac{\varphi(z)\overline{\varphi''(z)}}{\overline{\varphi'^2(z)}} - \overline{\psi'(z)} \right] \tag{1.4}$$

$$\sqrt{I} = \frac{\partial z^*}{\partial z} \frac{\partial \bar{z}^*}{\partial \bar{z}} - \frac{\partial z^*}{\partial \bar{z}} \frac{\partial \bar{z}^*}{\partial z}, \quad q = 2\sqrt{\frac{\partial z^*}{\partial z} \frac{\partial \bar{z}^*}{\partial \bar{z}}}, \quad \Omega(q) = q - \frac{2(\lambda + \mu)}{\lambda + 2\mu} \tag{1.5}$$

$$z^* = z + u + iv$$

It is proved /3/ that

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$$\varphi'(z) \neq 0 \text{ everywhere in } S^- + L \tag{1.6}$$

Moreover, for the case under consideration for large $|z|$

$$\varphi(z) = -\frac{(\lambda + 2\mu)(X + iY)}{4\pi\mu(\lambda + \mu)} \ln z + z + o(1) + \text{const} \tag{1.7}$$

$$\psi(z) = \frac{(\lambda + 2\mu)(X - iY)}{2\pi\mu(\lambda + \mu)} \left[\frac{1}{\overline{\varphi'}(z)} - 1\right] \ln z + o(1) + \text{const} \tag{1.8}$$

where (X, Y) is the principal vector of the external forces applied to L_1.

On the basis of (1.2) and (1.4) and the absence of stresses at infinity it follows from (1.1) that

$$\overline{\varphi(x)}\, \varphi''(x) - \varphi'^2(x)\, \psi'(x) = 0 \text{ on } L \tag{1.9}$$

Taking account of (1.9) we obtain from (1.2)-(1.6)

$$Y_y^- = N(x) = \frac{2\mu(\lambda + \mu)[|\varphi'^2(x)| - 1]}{\lambda + \mu + \mu|\varphi'^2(x)|} \text{ on } L \tag{1.10}$$

Hence, we will have according to the second condition of (1.1)

$$|\varphi'(x)| = \exp F(x) \text{ on } L_1, \quad |\varphi'(x)| = 1 \text{ on } L_2 \tag{1.11}$$

Here

$$F(x) = \frac{1}{2} \ln \left[\frac{\lambda + \mu}{\mu} \frac{2\mu + N(x)}{2(\lambda + \mu) - N(x)}\right] \tag{1.12}$$

Taking account of (1.6) and (1.7) we find from (1.11)

$$\varphi'(z) = \exp\left(-\frac{1}{\pi i} \int_{L_1} \frac{F(x)\, dx}{x - z}\right), \quad z \in S^- \tag{1.13}$$

We now find the boundary values of $\varphi'(z)$ on L from S^- and we insert the expression obtained into (1.3). Then taking account of (1.9), the last condition in (1.1), and (1.12), we obtain the fundamental relationship establishing the non-linear relation between the desired functions $F(x)$ and $u(x)$ on L_1

$$\int_{L_1} \frac{F(x)\, dx}{x - x_0} = \frac{\pi}{2} \arcsin \left[\left(1 - \frac{N(x_0)}{2(\lambda + \mu)}\right)(1 + u'(x_0)) f'(x_0 + u)\right], \quad x_0 \in L_1 \tag{1.14}$$

It can be shown that the expression in the square brackets on the right side of (1.14) does not exceed unity in absolute value.

Indeed, we insert the boundary value of the function (1.13) from S^- on L into the formula following from (1.3) and (1.9)

$$u'(x) = \left(\frac{\mu}{\lambda + 2\mu} + \frac{\lambda + \mu}{\lambda + 2\mu} \frac{1}{|\varphi'^2(x)|}\right) \operatorname{Re} \varphi'^2(x) - 1 \text{ on } L_1$$

and we take account of (1.14) in the expression obtained. We then obtain the relationship

$$\frac{N(x)}{2(\lambda + \mu)} = 1 - \frac{1}{(1 + u'(x))\sqrt{1 + f'^2(x + u)}} \text{ on } L_1 \tag{1.15}$$

from which the required estimate follows.

The equalities (1.12), (1.14) and (1.15) form a system of three functional equations to determine the functions $F(x)$, $N(x)$ and $u(x)$ on L_1.

Taking acccount of (1.15) on the right-hand side of (1.14) we obtain

$$\int_{L_1} \frac{F(x)\, dx}{x - x_0} = \frac{\pi}{2} \arcsin \frac{f'(x_0^*)}{\sqrt{1 + f'^2(x_0^*)}} \quad (x_0^* = x_0 + u) \tag{1.16}$$

where the real function $f(x_0^*)$ characterizes the shape of the stamp base in the deformed state. But on the other hand, the expression under the arcsin equals $\sin\alpha(x_0^*)$, where $\alpha(x_0^*)$ is the angle that the tangent drawn at the point $(x_0^*, f(x_0^*))$ makes with the positive direction of the real axis. Therefore

$$\int_{L_1} \frac{F(x)\,dx}{x - x_0} = \frac{\pi}{2}\alpha(x_0^*) = \frac{\pi}{2}\alpha(x_0 + u) \tag{1.17}$$

Considering the right-hand side of (1.17) to be known temporarily, we arrive at a homogeneous characteristic singular integral equation to determine the function $F(x)$ on L_1. Below we shall seek a solution of this equation in the class h_0 (i.e., a solution not bounded at the ends of the line of integration). The index of this class equals one, and the solution itself has the form (without loss of generality we will assume that $L_1 = [-a; a]$) /5/

$$F(x_0) = \frac{1}{2\pi\sqrt{a^2 - x_0^2}} \int_{-a}^{a} \frac{\alpha(x^*)\sqrt{a^2 - x^2}\,dx}{x - x_0} + \frac{C}{\sqrt{a^2 - x_0^2}} \tag{1.18}$$

where C is a real constant determined from the condition for specifying the principal vector of the acting stresses.

After having determined the function $F(x)$ we find the function $\varphi'(z)$ from (1.13), and we determine the other desired potential $\psi(z)$ from (1.9) by a well-known method. According to the function $F(x)$ found the contact pressure distribution under the stamp is determined in conformity with (1.12) by the expression

$$N(x) = \frac{2\mu\,[\exp(2F(x)) - 1]}{1 + \mu(\lambda + \mu)^{-1}\exp(2F(x))} \tag{1.19}$$

It should be noted that the function $\alpha(x^*)$ on the right-hand side of (1.17) is generally unknown, which complicates the investigation considerably. A case is examined below when this obstacle is overcome successfully and the exact solution of the problem is found.

2. ***A stamp with a rectilinear horizontal or inclined base.*** If the stamp has a rectilinear base with angle of inclination ω, then $\alpha(x^*) = \omega$. Since

$$\int_{-a}^{a} \frac{\sqrt{a^2 - x^2}\,dx}{x - x_0} = -\pi x_0$$

we find the solution of class h_0 of Eq.(1.17) from (1.18) in the form

$$F(x) = \frac{\omega x + C}{2\sqrt{a^2 - x^2}}, \qquad C = \frac{(\lambda + 2\mu)N_0}{4\pi\mu(\lambda + \mu)} \tag{2.1}$$

Here N_0 is a given positive constant, $Y = -N_0$, and the constant C is determined as a result of integrating the first equality in (2.1) between $-a$ and a, comparing the expression obtained with the asymptotic value of the right-hand side of (1.13) for large $|z|$ and taking account of (1.7).

Therefore, the contact pressure under the stamp will be determined by (1.12) and (2.1), where the case $\omega = 0$ corresponds to a stamp with a rectilinear horizontal base.

Thus, the normal stress distribution has been obtained in the contact domain (including the corner points) that contains no singularities. In particular $\lim N(x) = 2(\lambda + \mu)$ as $|x| \to a$. Moreover, the distribution obtained depends very much on the elastic properties of the material.

The principal moment of the external forces maintaining the stamp in a given position can be evaluated from the formula

$$M = -\int_{-a}^{a} N(x)\,dx$$

3. ***A stamp with a rectilinear wedge-shaped base.*** Let the stamp have a wege-shaped base, i.e., be a broken line symmetrical about the y axis with apex at the point $x = 0$. The slope of this line equals $|\alpha|$. It is assumed that finite (corner) points of the stamp $(x = \pm a)$ make contact with the elastic half-plane boundaries. We have

$$d(x^*) = \begin{cases} -\alpha \text{ on } [-a; 0[\\ \alpha \text{ on }]0, a] \end{cases}$$

Substituting this result into the right-hand side of (1.18), we obtain

$$F(x) = -\frac{\sigma}{\pi} \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right| + \frac{(\lambda + 2\mu) N_0 + \mu(\lambda + \mu)\alpha a}{2\pi\mu(\lambda + \mu)\sqrt{a^2 - x^2}} \tag{3.1}$$

and according to (1.19) we will have the contact pressure distribution pattern. The contact pressure will have the finite value $2(\lambda + \mu)$ at the points $a, 0, a$

This example yields a definite representation of the process of impressing a cutting instrument into an elastic medium.

4. On the bounded solutions of (1.16). We consider the original Eq.(1.16) and we determine the solution of this equation of the class $h(-a, a)$ (i.e., the solution bounded at the points $-a, a)$, where it is assumed that $F(\pm a) = 0$ Using this latter condition we differentiate (1.16) with respect to x_0. We obtain

$$\int_{-a}^{a} \frac{F'(x)\,dx}{x - x_0} = \frac{\pi}{2} \frac{k(x_0^*)}{\cos\alpha} \tag{4.1}$$

where $k(x_0^*)$ is the curvature of the contact line at the point x_0^*, and α is the angle made by the tangent to this line at the point $(x_0^*, f(x_0^*))$ with the positive direction of the real axis.

We now assume that the stamp is a strip bounded from below by the symmetric arc of a circle of radius R and on the side by the vertical lines $x = \pm a$ Then $k(x^*) = 1/R$, where R is a sufficiently large quantity. The right side of (4.1) can be replaced by the expression $\pi/(2R)$ with acceptable accuracy, i.e., components of third order of smallness in $1/R$ can be neglected. This means we will have

$$\int_{-a}^{a} \frac{F'(x)\,dx}{x - x_0} = \frac{\pi}{2R} \tag{4.2}$$

We now seek the solution of this equation of class h_0. It has the form /6/

$$F'(x) = \frac{A + Bx}{\sqrt{a^2 - x^2}}$$

where A and B are real constants. We integrate this equality and take into account the symmetry condition of the problem. We then obtain $(A = 0)$

$$F(x) = C\sqrt{a^2 - x^2}, \quad C = (2R)^{-1} \tag{4.3}$$

The constant C is determined by substituting the first expression in (4.3) into (4.2).

To determine the real constant a we substitute (4.3) into the right-hand side of (1.13) and we compare the asymptotic behaviour of (1.7) and (1.13) for large $|z|$. We then obtain

$$a = \{(\lambda + 2\mu) R N_0/[\pi\mu(\lambda + \mu)]\}^{1/2}$$

The half-length of the contact section is determined by this formula.

According to (1.19) and (4.3) the contact pressure under the stamp vanishes at the end points of the contact domain.

Finally, we consider the case when the stamp has a wedge-shaped base with slope $|\alpha|$ and the acting external forces are not sufficient for the stamp corners to come into contact with the elastic half-plane boundaries. Therefore, the ends of the contact line (i.e., the points $-a, a)$ are not known in advance and must be determined when solving the problem.

In the case under consideration the function $F(x)$ is determined by the first (logarithmic) component on the right-hand side of (3.1) and we find the coordinate of the end point of the domain in the form

$$a = (\lambda + 2\mu) N_0/[2\mu(\lambda + \mu)\alpha]$$

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THE CONSTRUCTION OF THE DISSIPATIVE PLASTIC FLOW FUNCTION ON THE BASIS OF MICROSCOPIC REPRESENTATIONS*

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A dissipative function (DF) of the plastic flow of a single crystal is constructed on the basis of microscopic representations. A thermodynamic analysis is performed of the possible mechanical energy dissipation mechanism for moving dislocations. The general expression constructed for the DF is reduced to a form such that the latter depends only on characteristics of the process (strain rates) and macroscopic characteristics of the ensemble of dislocations. The physical meaning is uncovered here and the value of all the coefficients in the determination of the DF is indicated. The deduction is made that the phenomenological representation of the DF just as the sum of first and second degree homogeneous functions in the plastic strain rates is generally non-uniform and the rate of change of the mocrostructure parameters must still be taken into account.

The construction of the dissipative function (DF)

$$\Phi = T^{-1}dq'/dt \tag{0.1}$$

(where T is the absolute temperature, q' is the uncompensated heat, and t is the time) governing the magnitude of entropy growth due to internal irreversible processes is the most important element in describing plastic deformation and the construction of new models of continuous media /1, 2/. Usually it is postulated phenomenologically that the DF for plastic media is a homogeneous (linear or non-linear) function of first degree in the plastic strain rates ε_{ij} while it is a homogeneous second-degree function in the plastic strain rates or the sum of the above-mentioned first and second degree homogeneous functions for viscoplastic media /3, 4/. It is impossible to regard such an approach as completely satisfactory for the following reasons. 1°. It is assumed that the coefficients in the determination of the homogeneous functions can be determined experimentally. As a rule, however, the appropriate experimental data have a large spread. 2°. The coefficients mentioned are not determined from physical representation, i.e., on the basis of the material microstructural characteristics, whereupon their physical meaning is also not clear. 3°. It is also not known whether a DF of a plastic medium with dislocations can be constructed just like a homogeneous function (or the sum of homogeneous functions) in the plastic strain rates without taking account of the microstructural parameters of the plastic flow and their rate of change.

The DF was introduced in the form of the general expression (/5-7/, et al.)

$$\Phi = \Phi(\varepsilon_{ij}^{\cdot}, T, \mu, \mu^{\cdot})$$

when taking account of the internal parameters and their rates of change, where μ is an internal parameter. However, the form of the DF is not made specific here and the proposed models were not properly microscopic.

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